

NOTES ON UNIFORM EXPONENTIAL GROWTH AND TITS ALTERNATIVE

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These notes contain results concerning uniform exponential growth which were obtained in collaborations with E. Breuillard and A. Salehi-Golsefidy, mostly during 2005, improving Eskin-Mozes-Oh theorem [7], as well as a uniform version of Tits alternative improving [6].

1. MAIN STATEMENTS ABOUT ENTROPY

Let Σ be a subset of a group Γ . The algebraic entropy of Σ is defined to be

$$h(\Sigma) := \lim_{n \rightarrow \infty} \frac{1}{n} (\Sigma \cup 1)^n.$$

If $h(\Sigma) > 0$, we say that Σ has exponential growth. If $\inf\{h(\Sigma) : \Gamma = \langle \Sigma \rangle\} > 0$, we say that Γ has uniform exponential growth. We denote by $d^+(\Sigma) \in \mathbb{N} \cup \{\infty\}$ the minimal integer n , if such exists, for which $(\Sigma \cup \{1\})^n$ contains two elements a, b which generates a free semigroup. One of the main result proved in this note is:

Theorem 1.1. *For any integer $d > 1$ there is a constant $m = m(d) > 1$ such that for any field \mathbb{F} and any finite subset $\Sigma \subset GL_d(\mathbb{F})$ which generates a non-virtually solvable group we have $d^+(\Sigma) \leq m$. It follows that there is a constant $C = C(d) > 0$ such that $h(\Sigma) > C$ for any such S .*

The constants m, C can be effectively estimate, but in order to keep this note short and simple we will allow ourselves to use some compactness arguments which will make the proofs non effective. Theorem 1.1 answers a question of Gromov. It improves the result of [7] in the following three senses:

- (1) The analog result in [7] was proved only for symmetric sets.
- (2) In [7] it is assumed that the characteristic of \mathbb{F} is 0.
- (3) The constants m, C in Theorem 1.1 do not depend on the actual group generated by Σ , but only on the dimension d .

Remark 1.2. Uniform exponential growth for solvable non-virtually nilpotent groups was proved independently by Osin and by Wilson. The existence of a uniform constant $C(d)$ as in Theorem 1.1 for solvable groups is very striking since it implies the well known Lehmer conjecture concerning the Mahler measure of algebraic integers. However for virtually solvable non-virtually nilpotent groups we only obtained the analog uniform result for discrete subgroups of $GL_d(k)$ where k is a local field. The details of this result will appear elsewhere.

Using first order logic, it is straightforward to show that Theorem 1.1 is equivalent to the following result concerning finite groups:

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Theorem 1.3. *Given $d > 1$, there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ tending to infinity, such that for any finite simple group G of Lie type of dimension d , and any generating set Σ of G containing the identity, there are two elements $a, b \in \Sigma^{m(d)}$ such that the directed Cayley graph of G with respect to $\{a, b\}$ has girth $\geq f(|G|)$.*

2. EIGENVALUES VS. NORMS

Let k be a local field with absolute value $|\cdot|_k$. It induces the standard norm on k^d which in turn gives rise to an operator norm $\|\cdot\|_k$ on $M_d(k)$. If k is not Archimedean, let \mathcal{O}_k be its ring of integers and m_k the maximal ideal in \mathcal{O}_k . We note that $\|a\|_k \geq 1$ for all $a \in \mathrm{SL}_d(k)$. Let $\Lambda_k(a)$ be the maximum absolute value of all eigenvalues of a (recall that the absolute value has a unique extension to the algebraic closure of k).

For a compact subset $Q \subset M_d(k)$ we denote:

$$\begin{aligned} \Lambda_k(Q) &= \max\{\Lambda_k(a) : a \in Q\} \\ \|Q\|_k &= \max\{\|a\|_k : a \in Q\} \\ \Delta_k(Q) &= \inf_{g \in \mathrm{GL}_d(k)} \|gQg^{-1}\|_k. \end{aligned}$$

We will make use of the following strengthening of Lemma 4.2 from [6]:

Lemma 2.1. *There exists a constant $c = c(d)$ such that for any compact subset $Q \subset M_d(k)$ we have*

$$\Lambda_k(Q^i) \geq c \cdot \Delta_k(Q)^i$$

for some $i \leq d^2$. Moreover, if k is non-Archimedean the inequality holds with $c = (|\pi|_k)^{2d-1}$ for a uniformizer π for k .

The proof of Lemma 2.1 relies on the following two lemmas. We refer to [6] for proofs of Lemmas 2.1 and 2.2.

Lemma 2.1. *Let R be a field or a finite ring and let $\mathcal{A} \leq M_d(R)$ be a subring and R -submodule. Suppose that \mathcal{A} is spanned as an R -module by nilpotent matrices, then \mathcal{A} is nilpotent, i.e. $\mathcal{A}^n = \{0\}$ for some $n \geq 1$.*

Lemma 2.2. *For a compact subset $Q \subset M_d(k)$ the following are equivalent:*

- (i) Q generates a nilpotent subalgebra.
- (ii) $\Delta_k(Q) = 0$.
- (iii) $\Lambda_k(Q^i) = 0, \forall i \leq d^2$.

Proof of Lemmea 2.1. Suppose by contradiction that there is a sequence of compact sets Q_1, Q_2, \dots in $M_d(k)$ such that $\Lambda_k(Q_n^i) < \Delta_k(Q_n^i)/n, \forall i \leq d^2$. By replacing Q_n with a suitable conjugate of it, we may assume that $\|Q_n\|_k \leq 2\Delta_k(Q_n)$, and by normalizing we may assume that $\|Q_n\|_k = 1$. Let Q be a limit of Q_n with respect to the Hausdorff topology on $M_d(k)$. Then $\|Q\|_k = 1$, $\Delta_k(Q) \geq \frac{1}{2}$ since Δ_k is upper semi-continuous, and by continuity of Λ_k , $\Lambda_k(Q^i) = 0, \forall i \leq d^2$. This however contradicts Lemma 2.2.

Let us now explain why one can take $c = (|\pi|_k)^{2d-1}$ in the non archimedean case. Let \mathcal{O}_k be the ring of integers of k and m_k its maximal ideal. Suppose by contradiction that there is a compact subset Q of $M_d(k)$ such that $\Lambda_k(Q^i) < c \cdot \Delta_k(Q)^i$ for every $i \leq d^2$. Then up to conjugating by a suitable element of $\mathrm{SL}_d(k)$ and renormalizing, we may assume that $\Delta_k(Q) = \|Q\| = 1$. Note

that since $\Lambda_k(Q^i) \leq (|\pi|_k)^{2d}$, the coefficients of the characteristic polynomial of any element of Q^i belong to m_k^{2d} . Let \mathcal{A} be the ring and \mathcal{O}_k -submodule generated by Q in $M_d(\mathcal{O}_k)$. Let $\overline{\mathcal{A}}$ its image modulo m_k^{2d} in $M_d(\mathcal{O}_k/m_k^{2d})$. Then $\overline{\mathcal{A}}$ is spanned as an \mathcal{O}_k/m_k^{2d} -module by nilpotent matrices. Hence, according to Lemma 2.1, $\overline{\mathcal{A}}$ is nilpotent and therefore lies in the upper triangular matrices after a suitable change of basis. Hence $\mathcal{A} \leq m_k^{2d}M_d(\mathcal{O}_k) + N_d(\mathcal{O}_k)$, where $N_d(\mathcal{O}_k)$ are the upper triangular matrices with 0 diagonal and coefficients in \mathcal{O}_k . However, we can now conjugate \mathcal{A} by the matrix $t = \text{diag}(\pi^{-(d-1)}, \pi^{-d-1}, \dots, \pi^{d-1}) \in SL_d(k)$ and get that $t\mathcal{A}t^{-1} \leq m_k^2M_d(\mathcal{O}_k)$, contradicting that $\Delta_k(Q) = 1$. \square

Remark 2.3. Note that in the non archimedean case, if we had considered the algebraic closure \overline{k} of k instead of k , then the constant c could have been taken to be equal to 1. Indeed, if k_n denotes the compositum of all extensions of k of degree at most n , and π_n a uniformizer for k_n , then $|\pi_n|_{\overline{k}}$ tends to 1 as n tends to infinity.

3. LOWER BOUND ON THE ALGEBRAIC ENTROPY

In this section, we prove the following theorem.

Theorem 3.1. *Let G be a Zariski connected simple group over a local field k , considered via its Adjoint representation as an irreducible subgroup of $SL(\mathfrak{g})$, and let $\delta > 0$. Then there is a constant $m = m(\dim G, \delta, k)$ such that for any compact subset Q generating a Zariski dense subgroup, which is assumed to be non-compact in the non-Archimedean case, with $\Delta_k(Q) \geq 1 + \delta$, we have $d^+(Q) \leq m$.*

We shall require a few lemmas.

Lemma 3.2. *For any C and $\delta > 0$ there is $N(C, \delta, d)$ such that if $Q \subset SL_d(k)$ is a compact set with $\Delta_k(Q) \geq 1 + \delta$ then $\|\cup_{i=1}^N Q^i\|_k > C$.*

Proof. Let us first assume that k is Archimedean. Assuming the contrary, we obtain a sequence of compact sets Q_n with $\Delta_k(Q_n) \geq 1 + \delta$ and $\|\cup_{i=1}^n Q_n^i\| \leq C$. Since the norms of Q_n are uniformly bounded we can pass to a subsequence which converges to a limit compact set Q , satisfying:

$$\|\cup_{i=1}^{\infty} Q^i\| \leq C.$$

It follows that the closed semigroup generated by Q is compact. Since a compact sub-semigroup of a topological group is a group, we obtain that $\Delta_k(Q) = 1$, contradicting the assumption that $\Delta_k(Q_n) \geq 1 + \delta$, as Δ_k is upper semi continuous.

Now if k is non-Archimedean we can argue in the same way, assuming further that $\langle Q_n \rangle$ are non-compacts, and berrying in mind that any compact subgroup is contained in an open compact subgroup. \square

Lemma 3.3. *Suppose $a \in SL_d(k)$ satisfies $\Lambda_k(a) \geq 2\lambda_k(a)$ where $\lambda_k(a)$ is the modulus of the second highest eigenvalue of a . Then the top eigenvalue α_1 belongs to k , $|\alpha_1| = \Lambda_k(a)$ and there exists $h \in SL_d(\overline{k})$ with $\|h\| \leq 6^d \|a\|^{2d}$ such that the matrix $a' = hah^{-1}$ is such that $a'(e_1) = \alpha_1 e_1$ and $a'(H) = H$ where $H = \langle e_2, \dots, e_d \rangle$ and $\|a'|_H\| \leq \frac{3}{2}\lambda_k(A)$.*

Let us first prove:

Sublemma 3.4. *For any $a \in SL_d(k)$ there exists $h \in SL_d(\overline{k})$ with $\|hah^{-1}\| \leq \frac{3}{2}\Lambda(a)$ and $\|h\| \leq 2^{d-1} \|a\|^{d-1}$.*

Proof. We can triangularize a , i.e. find a basis u_1, \dots, u_d of \bar{k}^d such that $a(u_i) = \lambda_i u_i +$ other terms in $\text{span}\{u_j, j > i\}$. Then we can “orthonormalize” this basis, i.e. we can find g in the standard maximal compact subgroup of $SL_d(\bar{k})$ (i.e. $\|g\| = 1$) such that $u_i = \mu_i g e_i +$ other terms in $\text{span}\{u_j, j > i\}$, where $\mu_i \in \bar{k}$ and e_1, \dots, e_d is the canonical basis of \bar{k} . Hence up to conjugating a by g^{-1} (this doesn’t change the norm) and rescaling each u_i we may assume that we had started with a triangular with coefficients in \bar{k} .

Now we can take $h = t^{\frac{d+1}{2}} \text{diag}(t^{-1}, \dots, t^{-d}) \in SL_d(\bar{k})$. Let $a = (a_{ij})$, then $\max |a_{ij}| \leq \|a\|$. We get $\|hah^{-1}\| \leq \Lambda(a) + 2t^{-1} \|a\|$. Indeed for $x \in \bar{k}$ we have $hah^{-1}x = \sum \lambda_i x_i e_i + \sum_{j>i} x_i t^{i-j} a_{ij} e_j$. Let us estimate the second term (say in the archimedean case):

$$\left\| \sum_{j>i} x_i t^{i-j} a_{ij} e_j \right\|^2 = \sum_{1 \leq i, i' \leq d-1} x_i \bar{x}_{i'} \sum_{j>i, i'} t^{i+i'-2j} a_{ij} \bar{a}_{i'j} \leq \|a\|^2 \sum_{1 \leq i, i' \leq d-1} |x_i| |x_{i'}| \sum_{j>i, i'} t^{i+i'-2j}.$$

However $\sum_{j>i, i'} t^{i+i'-2j} \leq t^{i+i'} \frac{t^{-2(\max\{i, j\}+1)}}{1-t^{-2}} \leq 2t^{-2} t^{-|i-i'|}$ if $1 - t^{-2} \geq 1/2$. So

$$\left\| \sum_{j>i} x_i t^{i-j} a_{ij} e_j \right\|^2 \leq 2t^{-2} \|A\|^2 \sum_{1 \leq i, i' \leq d-1} |x_i| |x_{i'}| t^{-|i-i'|}.$$

The right hand side is the quadratic form associated to the matrix $B := (t^{-|i-i'|})_{i, i'}$. We compute easily $\|B\| \leq 1 + \frac{2t^{-1}}{1-t^{-1}} \leq 2$ if $t \geq 3$. Hence

$$\left\| \sum_{j>i} x_i t^{i-j} a_{ij} e_j \right\|^2 \leq 4t^{-2} \|A\|^2 \|x\|^2.$$

Thus we can take $t = 4\|a\|/\Lambda(a)$ in case k is archimedean. When k is non-archimedean, we can take $t = \pi^{-n}$, where $|\Lambda(a)| = |\pi^{n-1} \|a\||$ (so that $|t| \leq \|a\|^2/\Lambda(a)^2$). Hence in both cases we get $\|hah^{-1}\| \leq \frac{3}{2}\Lambda(a)$ and $\|h\| \leq t^{\frac{d-1}{2}} \leq 2^{d-1} \|a\|^{d-1}$, since $\Lambda(a) \geq 1$ because $a \in SL_d(\bar{k})$. \square

Proof of Lemma 3.3. As in the proof of sublemma 3.4, we can conjugate a to some lower triangular matrix without changing the norm of a . Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be the eigenvalues of a in \bar{k} , where we chose $|\alpha_1| = \Lambda_k(a)$ and $|\alpha_2| = \lambda_k(a)$. Note that since $\Lambda_k(a) > \lambda_k(a)$ all Galois conjugates of α_1 (whose modulus is the same as that of α_1) must be equal to α_1 itself. Hence $\alpha_1 \in k$. Let H be the hyperplane in \bar{k}^d spanned by e_2, \dots, e_d and let $v = (1, x_2, \dots, x_d)$ be the eigenvector corresponding to α_1 . We claim that $d(v, H) \geq 1/(1+L)^{d-1}$, where $L = 2\|a\|/\Lambda_k(a)$.

Indeed we have $d(v, H) \geq 1/\sqrt{1 + \sum |x_i|^2}$ with equality in the archimedean case. But if $(\alpha_1, a_2, \dots, a_d)$ is the first column of a , then $a_2 + \alpha_2 x_2 = \alpha_1 x_2$ hence $|x_2| \leq |a_2|/|\alpha_1 - \alpha_2| \leq 2\|a\|/\Lambda_k(a)$. Then similarly we get $|x_3| \leq L + L^2$ and finally by induction $|x_k| \leq L(1+L)^{k-2}$. We compute $1 + \sum |x_i|^2 \leq (1+L)^{2d-2}$, hence $d(v, H) \geq 1/(1+L)^{d-1}$.

Now let $h_1 \in SL_d(\bar{k})$ be defined by $h_1 v = e_1$ and $h_1 e_i = e_i$ for $i > 1$. It is easy to check directly in both archimedean and non-archimedean case that $\|h_1\|^2 \leq 2(1 + \sum |x_i|^2)$, hence $\|h_1\| \leq \sqrt{2}(1+L)^{d-1}$.

Finally, we can apply Lemma 3.4 to the restriction of a to H to get $h_0 \in SL_d(\bar{k})$ with $\|h_0\| \leq 2^{d-2} \|a\|^{d-2}$ and $\|h_0 a|_H h_0^{-1}\| \leq \frac{3}{2} \lambda_k(a)$. Finally letting $h = h_0 h_1$ and $a' = hah^{-1}$, we have $a' e_1 = \alpha_1 e_1$, $a' H = H$, $\|a'\|_H \leq \frac{3}{2} \lambda_k(a)$ and $\|h\| \leq \|h_0\| \|h_1\| \leq 6^d \|a\|^{2d}$. \square

The following lemma is verified by a direct simple computation. We refer the reader to [3] Section 3, for a detailed study of the dynamics of projective transformations.

Lemma 3.5. *Let $a \in SL_d(k)$ and assume that $a(e_1) = \alpha e_1$, $a(H) = H$, where H is the hyperplane spanned by e_2, \dots, e_d , and $\|a|_H\| \leq \epsilon^2 |\alpha|$, for some $\epsilon \leq 1/4$. Then $[a]$ is ϵ proximal with respect to $([e_1], [H])$, i.e. it takes the complement of the ϵ -neighborhood of $[H]$ into the ϵ -neighborhood of $[e_1]$. Moreover a is ϵ -Lipschitz outside the ϵ -neighborhood of $[H]$.*

The following is a variant of the classical ping-pong lemma.

Lemma 3.6 (The Ping lemma). *Let a be as in Lemma 3.5, and let $b \in SL_d(k)$ be another element with $b(e_1) \neq e_1$, $(\|b\| \cdot \|b^{-1}\|)^2 \leq \epsilon^{-\frac{1}{2}} - 1$ and $d(b \cdot [e_1], [H]) \geq \sqrt{\epsilon}$. Then a and ba generate a free semigroups.*

Proof. Let $B_\epsilon([e_1])$ denote the open ϵ ball around $[e_1]$. The conditions on b implies that $b(B_\epsilon([e_1]))$ does not meet the ϵ -neighborhood of $[H]$, indeed by Lemma 3.1 in [3] b acts on \mathbb{P}^{d-1} with Lipschitz constant $(\|b\| \|b^{-1}\|)^2$. Let

$$\tau = \sup\{t \leq \epsilon : b(B_t([e_1])) \cap B_t([e_1]) = \emptyset\}.$$

Then $\tau > 0$ and if we put $U = B_\tau([e_1])$ and $V = b \cdot U$, then $U \cap V = \emptyset$, $a \cdot (U \cup V) \subset U$ and $ba \cdot (U \cup V) \subset V$. \square

We shall also make use of the following:

Lemma 3.7. *Let G and Q be as in Theorem 3.1 and consider an irreducible representation ρ of G on k^d . Then there is an element $q \in Q^{d+1}$ such that $\rho(q) \cdot [e_1] \neq [e_1]$ and $\rho(q) \cdot [e_1] \notin [H]$.*

Proof. As G is Zariski connected ρ is strongly irreducible on k^d and hence there are $d-1$ elements in $Q^{d-1} \cdot [e_1]$ which together with e_1 form a basis to k^d . Denote the corresponding projective points by $[e_1], q_1 \cdot [e_1], \dots, q_{d-1} \cdot [e_1]$ where $q_i \in Q^{d-1}$. Since G is Zariski connected it has no non-trivial map to the symmetric group on d elements and hence for some $q_d \in Q^d$ the cardinality of the set $\{[e_1], q_1 \cdot [e_1], \dots, q_d \cdot [e_1]\}$ is $d+1$. If for some $i \leq d$ the point $q_i \cdot [e_1]$ is not in $[H]$ then we are done. Assume that this is not the case, then one can easily verify that our points $q_i \cdot [e_1]$, $i = 1, \dots, d$ are in general position in $[H]$, i.e. the lines corresponding to any $d-1$ of them spans H . Let $q_0 \in Q$ be any element such that $q_0 \cdot [H] \neq [H]$. Then for some $i \neq j \leq d$, both $(q_0 q_i) \cdot [e_1]$ and $(q_0 q_j) \cdot [e_1]$ are not in $[H]$, and at least one of them is different from $[e_1]$. \square

Equipped with the lemmas above we can now prove Theorem 3.1:

Proof of Theorem 3.1. We assume $d = \dim G$ and view G as an irreducible subgroup of $SL_d(k)$. We may assume that $1 \in Q$ and hence $Q^j \supset Q$ for any integer j , and in particular Q^j generates a Zariski dense subgroup.

Let c be the constant from Lemma 2.1, let $N = N(2^{d-1}/c, \delta, d)$ be the constant from Lemma 3.2. Replacing Q by Q^N we may assume that $\Delta_k(Q) \geq 2^{d-1}/c$.

Fix $i \leq d^2$ as in Lemma 2.1, so that $\Lambda_k(Q^i) \geq c \Delta_k(Q^i)$ and take $a \in Q^i$ with $\Lambda_k(a) = \Lambda_k(Q^i)$. Then

$$\Lambda_k(a) \geq c \Delta_k \Delta_k(Q^i) \geq 2^{d-1},$$

and up to taking a suitable conjugation we may also assume $\Lambda_k(a) \geq \frac{c}{2} \|Q^i\|$.

Taking a suitable irreducible factor of some wedge representation (of degree at most d) we may assume that the top eigenvalue α_1 of a is unique and that $|\alpha_1/\alpha_2|_k \geq \Lambda_k(a)^{\frac{1}{d-1}} \geq 2$ where α_2 is second highest eigenvalue of a , so we can use Lemma 3.3 and find a suitable conjugating element $h \in \mathrm{SL}_D(k)$, where D is the dimension of the new representation. Note that the norm of a group element might have increased by at most a power d when we took the wedge representation, hence the inequality above is replaced by

$$\|a\|_k \geq \Lambda_k(a) \geq (c\|Q^i\|)^{\frac{1}{d}}, \quad \text{additionally } \|h\| \leq 6^{D^2} \|a\|^{2D^2}.$$

After conjugating with h we may assume that $a(e_1) = \alpha_1 e_1$, $a(H^-) = H^-$, where $H^- = \mathrm{span}\{e_j : j \geq 2\}$ and

$$\Lambda_k(a)^d \geq \frac{c}{6^{2d^2}} \|Q^i\|^{\frac{1}{4D^5}}.$$

Let l be such that $|\alpha_1/\alpha_2|_k^l \geq \|Q^{d^2}\|$.

Let us first explain the proof under the assumption that Q is symmetric, and then give an alternative argument which holds also without this assumption. By lemma 3.7 we can find $q \in Q^d$ such that $q \cdot [e_1] \neq [e_1]$ and $q \cdot [e_1] \notin [H]$. It follows that the element $b = q^{-1}aq$ satisfies $b^n \cdot [e_1] \rightarrow q^{-1} \cdot [e_1]$ and in particular, no power of b stabilizes $[e_1]$. By the Cayley-Hamilton theorem, we have $\sum_{1 \leq j \leq D} \beta_j b^j = -1$ for some coefficients β_j such that $|\beta_j|_k \leq 2^D \Lambda_k(b)^D$. Hence for some $1 \leq j \leq D$, the first coordinate of $b^j e_1$ is at least $\frac{1}{D 2^D \Lambda_k(b)^D}$. It follows that $d([b^j e_1], H^-) \geq \frac{1}{D 2^D \Lambda_k(b)^D \|b^j\|}$. Since the norm of $\|b\|$ is bounded by $|\alpha_1/\alpha_2|^l$ we can take a bounded power of a so that it will be ϵ -proximal for suitable ϵ so that Lemma 3.5 and Lemma 3.6 will hold for a, b .

Let us now explain how to carry the proof without the assumption that Q is symmetric. All we need is to show that there is an element b in some bounded power of Q such that $b^i \cdot [e_1] \neq [e_1]$, $\forall i \leq D$. The existence of such an element can be deduced from Lemma 3.8. \square

The following lemma from [6] is a generalization of the corresponding lemma from [7] and is proved by the same reasoning.

Lemma 3.8. *Given an integer χ there is $N = N(\chi)$ such that for any field K , any integer $d \geq 1$, any K -algebraic subvariety X in $GL_d(K)$ with $\chi(X) \leq \chi$ and any subset $\Sigma \subset GL_d(K)$ which contains the identity and generates a subgroup which is not contained in $X(K)$, we have $\Sigma^N \not\subset X(K)$.*

4. ENTROPY FOR DISCRETE SUBGROUPS

In this section we prove the following:

Theorem 4.1. *Given an integer $d > 1$, there is a constant $m' = m'(d)$ such that if $\Sigma \subset GL_d(\mathbb{R})$ is a finite set, generating a discrete non-virtually solvable group, then $d^+(\Sigma) \leq m'$.*

The proof relies on the classical Margulis lemma:

Lemma 4.2. *There exists a constant $\tau = \tau(d) > 0$ such that if $\Sigma \subset SL_d(\mathbb{R})$ is a finite set generating a discrete non-virtually nilpotent group, then $E_{\mathbb{R}}(\Sigma) \geq 1 + \tau$.*

The Margulis lemma is usually stated in terms of displacement, however the exact same argument (c.f. [10]) gives Lemma 4.2 with $\|\Sigma\|$ on the left side, and since discreteness is preserved under conjugation, we may replace $\|\cdot\|$ by $E_{\mathbb{R}}$. Note also that the analog statement hold for any local field, but, as remarked above, we do not need it. We will assume in this section that $\Gamma = \langle \Sigma \rangle$ is not virtually solvable. Let us first reduce to the case of simple Zariski closure.

Lemma 4.3. *Let Γ be a discrete non-virtually solvable linear group over k . Then Γ admits a representation ρ with discrete image whose Zariski closure is semisimple without compact factors.*

Proof. Let $G = \overline{\Gamma}^Z$ be the Zariski closure of Γ and let A be the amenable radical of G , i.e. the maximal closed normal amenable subgroup. Then G/A is semisimple without compact factors. Let ρ be the restriction of the quotient map $\rho : \Gamma \rightarrow G/A$. Since Γ is discrete in G , the action of Γ on G by left multiplications is amenable, and since A is amenable the action of Γ on G/A is also amenable.

In case $k = \mathbb{R}$ it follows from Zimmer's theorem (c.f. [4]) that the identity component of $\rho(\Gamma)$ is solvable, and since it is also normal, as $\rho(\Gamma)$ is Zariski dense, it must be trivial by the maximality of A , hence $\rho(\Gamma)$ is discrete.

Assume now that k is non-Archimedean and, by a way of contradiction, assume that $\rho(\Gamma)$ is not discrete. By the non-connected version of Zimmer's theorem (the generalized Connes-Sullivan conjecture, c.f. [4]), $\rho(\Gamma)$ contains an open solvable subgroup Γ_0 . Let U_0 be its closure, and let $U_n \subset U_0$ be a decreasing sequence of open compact subgroups in G/A with $\cap U_n = 1$. Our assumption implies that $U_n \cap \Gamma$ is non-trivial for every n . By Noetherity of the Zariski topology the sequence $\overline{U_n \cap \Gamma}^Z$ must stabilize after finitely many steps, say m . It follows that $\overline{U_m \cap \Gamma}^Z$ is a non-trivial closed normal solvable subgroup, contradicting the maximality of A . \square

Proof of Theorem 4.1. Let us assume now that $G = \overline{\Gamma}^Z$ is semisimple. Replacing Γ by a subgroup of bounded index we may also assume that it is Zariski connected. Indeed, look at the Adjoint representation of G . The image of Γ is discrete since the kernel is the center, hence amenable. Moreover, the image of $\text{Ad}(G)$ is a subgroup of $\text{Aut}(\text{Ad}(G)^\circ)$ and $[\text{Aut}(\text{Ad}(G)^\circ) : \text{Ad}(G)^\circ]$ is bounded in terms of the number of factors of $\text{Ad}(G)^\circ$. Projecting to an appropriate factor of G we can also assume that it is simple. Hence we can apply Theorem 3.1. \square

5. ENTROPY OF UNBOUNDED GROUPS OVER NON-ARCHIMEDEAN FIELDS

In this section we prove the following theorem:

Theorem 5.1. *For any integer $d > 1$ there is a number $m'' = m''(d)$ such that for any non-archimedean local field k and any subset $\Omega \subset \text{GL}_d(k)$ generating an unbounded group whose Zariski closure is semisimple, we have $d^+(\Omega) \leq m''$.*

The proof is similar to that of Theorem 3.1, while instead of using the uniform gap δ we use the ultrametric inequality.

Suppose that $\Delta_k(\Omega) = \beta_k$. Since $\langle \Omega \rangle$ is non-compact, $\beta_k > 1$. By Lemma 2.1 we may replace k by some large algebraic extension k' of k so that the constant c is arbitrarily close to 1. We have to make sure however that $\Delta_k(\Omega)$ does not get arbitrarily close to 1 in this procedure. This follows from the fact that the affine building X_k associated to $\text{GL}_d(k)$ embeds as a convex subset of the affine building $X_{k'}$ of $\text{GL}_d(k')$, and as these spaces are CAT(0) the projection from $X_{k'}$ to X_k is 1-Lipschitz. Since Ω , as a subset of $\text{SL}_d(k)$, preserves X_k , it follows that its minimal

displacement $\min(\Omega) := \min\{d(x, g \cdot x) : g \in \Omega, x \in X_{k'}\}$ is attained in X_k . Since $\min(\Omega)$ and $\Delta(\Omega)$ are related by the following inequality (cf. [6], Lemma 4.5):

$$\Delta_k(\Omega) \leq \exp(\min(\Omega)) \leq \Delta_k^{\sqrt{d}},$$

we see that $\Delta_{k'}(\Omega)$ is bounded away from 1. Thus, replacing k by an appropriate large extension, we may assume that the constant c of Lemma 2.1 satisfies $c^{-2} < \beta := \beta_k$.

Assume that $1 \in \Omega$. Let us also assume as we may by replacing Ω by a subset of a bounded power of it, using Lemma 3.8, that all the elements in Ω are semisimple. As in the proof of Theorem 3.1 we may replace the representation by a suitable irreducible factor of an exterior product, and obtain a strongly irreducible representation of dimension $D \leq d^d$ and an element $a \in \Omega^{d^2}$ with $|\alpha_1/\alpha_2| \geq \beta^{\frac{1}{d-1}}$ where α_1, α_2 are the eigenvalues of a of highest and second highest absolute values. Replace Ω by a suitable conjugate, we may also assume that $\|(\Omega)\| \leq \beta^d$.

Lemma 5.2. *Let v be the normalized eigenvector corresponding to α_1 and H the hyperplane spanned by the other eigenvectors. Then $d([v], [H]) \geq \frac{1}{\beta^{d^2}}$.*

Proof. Denote by \mathcal{O}_k the ring of integers in k . By the Gram-Schmidt argument we can conjugate a by an element of $\mathrm{GL}_D(\mathcal{O}_k)$ and obtain a triangular matrix whose first entry is a_1 . Conjugating it further by a suitable diagonal matrix of norm $\leq \beta^{d^2}$ we get an upper triangular matrix $a^g = ga g^{-1}$ for which all the entries other than the first are of absolute value $< |a_1|$. This implies that $gH = \mathrm{span}\{e_i : i \geq 2\}$, and hence the norm of the restriction $a^g|_{gH}$ is strictly less than $|a_1|$. This implies that $d([gv], [gH]) = 1$. Indeed, let $y \in gH$ be a normalized vector closest to $\hat{g}v$, and let $x = \hat{g}v - y$, then

$$|a_1| \cdot \|x\| = \|a^g\| \cdot \|x\| \geq \|a^g(x)\| = \|a^g(\hat{g}v) - a^g(y)\| = \|a^g(\hat{g}v)\| = |a_1|.$$

The lemma follows. \square

It follows that there is a matrix $h \in \mathrm{GL}_D(k)$ of norm 1 with inverse of norm $\leq \beta^{d^2}$ such that $h(e_1) = v$ and $h(\mathrm{span}\{e_i : i > 1\}) = H$.

As in the proof of Theorem 3.1, using the Cayley-Hamilton theorem, we find $b \in \Omega^D$ such that $h(a^{d^{10}})h^{-1}$ and $h(ba^{d^{10}})h^{-1}$ play "ping" with open attracting sets $B(\beta^{d^5}, [e_1])$ and $b^h B(\beta^{d^5}, [e_1])$. \square

6. UNIFORM ENTROPY FOR LINEAR GROUPS OVER ARBITRARY FIELDS

We shall now give the proof of Theorem 1.1. By a global field here we mean a finite algebraic extension of either \mathbb{Q} or $\mathbb{F}_q(t)$, where \mathbb{F}_q is the finite field with q elements and t is an indeterminate. Given a global field \mathbb{K} and a finite set S of places of \mathbb{K} including all the infinite ones, we denote by $\mathcal{O}_{\mathbb{K}}(S)$ the ring of S -integers in \mathbb{K} . The following lemma allows us to reduce the general case to Zariski dense subgroups of arithmetic groups.

Lemma 6.1 (cf. [6]). *Let \mathbb{F} be a field and let $\Gamma \leq \mathrm{GL}_d(\mathbb{F})$ be a finitely generated group which is not virtually solvable. Then there is a subgroup $\Gamma' \leq \Gamma$ of index $\leq d^2$, a global field \mathbb{K} , a finite set of places S of \mathbb{K} and a representation $f : \Gamma' \rightarrow \mathrm{GL}_d(\mathcal{O}_{\mathbb{K}}(S))$ whose image is Zariski dense in a simple \mathbb{K} -algebraic group.*

Moreover, it is easy to check that if Σ is a generating set for Γ then Σ^{2i+1} , where $i = [\Gamma : \Gamma']$, contains a generating set for Γ' .

In August 2006, Breuillard [2] announced the following beautiful Adelic version of the Margulis lemma:

Theorem 6.2. *For every integer $d > 1$ there is a constant $\delta = \delta_d > 0$ such that for every number field \mathbb{K} and a finite set $\Sigma \subset GL_d(\mathbb{K})$, which generates a group whose Zariski closure is simple, then either*

- *for some non-Archimedean completion k of \mathbb{K} the group $\langle \Sigma \rangle$ is unbounded, or*
- *for some Archimedean completion k of \mathbb{K} , $\Delta_k(\Sigma) \geq 1 + \delta$.*

Proof of Theorem 1.1. Let \mathbb{F} be a field and $\Sigma \subset GL_d(\mathbb{F})$ a finite set generating a non-virtually solvable group. By Lemma 6.1 it is enough to consider the case where $\Sigma \subset GL_d(\mathbb{K})$ for some global field \mathbb{K} and $\overline{\langle \Sigma \rangle}^z$ is connected and simple. Thus by Theorem 6.2 there is either a non-Archimedean completion k of \mathbb{K} where $\langle \Sigma \rangle$ is unbounded, or an Archimedean completion k of \mathbb{K} where $\Delta_k(\Sigma) \geq 1 + \delta(d)$. Thus Theorem 1.1 follows from Theorems 3.1 and 5.1. Note that in characteristic $p > 0$ the proof does not rely on Theorem 6.2. \square

7. UNIFORM TITS-ALTERNATIVE

The uniform version of Tits alternative proved in [6] state that for every given finitely generated non-virtually solvable subgroup $\Gamma \leq GL_d(\mathbb{F})$ there is a constant m_Γ such that for any symmetric subset Σ of Γ containing the identity which generates a Zariski dense subgroup (in particular every generating set), the m_Γ ball Σ^{m_Γ} contains two independent elements, i.e. two elements which generate a free non-abelian group. Our next claim is that the constant m_Γ could be made uniform for subgroups of GL_d , i.e. that there is a uniform constant m_d which applies for all non-virtually solvable subgroups of GL_d . In order to avoid some technical difficulties and to keep this note short and simple we will give the proof only for $d = 2$. The proof for general d will be given in [8].

Theorem 7.1. *There is a constant m_2 such that for any field \mathbb{F} and any finite symmetric set $\Sigma \subset SL_2(\mathbb{F})$ containing the identity which generate a Zariski dense subgroup, the set Σ^{m_2} contains two independent elements.*

Proof. By Lemma 6.1 we may replace \mathbb{F} by some global field \mathbb{K} .

Assume first that $\text{char}(\mathbb{K}) = 0$. In that case if $\mathbb{G} < \mathbb{SL}_2$ is a proper algebraic subgroup, then either $\dim(\mathbb{G}) = 0$ and by Jordan's theorem \mathbb{G} admits an abelian subgroup of bounded index, or $\dim(\mathbb{G}) = 1$ and \mathbb{G}° is unipotent and \mathbb{G} is solvable, or \mathbb{G}° is diagonalizable and $[\mathbb{G} : \mathbb{G}^\circ] \leq 2$, or $\dim(\mathbb{G}) = 2$ and \mathbb{G} is a Borel subgroup, hence solvable. Thus a non Zariski dense subgroup satisfies a simple law.

For any set A , denote by $A^{(2)}$ the set of squares $\{a^2 : a \in A\}$, and by A' the set of commutators $\{[a, b] : a, b \in A\}$. By Lemma 2.1 if $n \geq 4$ then $\Delta(\Sigma^n)^2 \geq \Delta((\Sigma^n)^{(2)}) \geq c\Delta(\Sigma^n)^2$ for any completion k of \mathbb{K} . By Lemma 3.8, up to replacing Σ by Σ^{n_0} for some constant n_0 , we may assume that $\Sigma^{(2)'} (the set of commutators of squares) does not satisfy the law from the previous paragraph, hence generates a Zariski dense subgroup. We then, using Theorem 6.2, chose an appropriate completion k of \mathbb{K} which is either non-Archimedean and $\langle \Sigma^{(2)'} \rangle$ is unbounded, or$

Archimedean and $\Delta_k(\Sigma^{(2)'}) \geq 1 + \delta$. Let us assume that k is Archimedean, the non-Archimedean case is treated similarly.

Using Lemmas 2.1 and 3.2 we get that for some constant n_1 , Σ^{n_1} contains a semisimple element a with eigenvalues $\alpha, \frac{1}{\alpha}$ with $|\alpha| \geq 100$, such that up to conjugating Σ by a suitable element, using Lemma 3.3, $a = \text{diag}(\alpha, \frac{1}{\alpha})$ and $\|\Sigma\| \leq |\alpha|^{f_1}$ for some constant f_1 . Set

$$d := \min\{\max\{|\sigma_{12}| : \sigma \in \Sigma^{(2)}\}, \max\{|\sigma_{21}| : \sigma \in \Sigma^{(2)}\}\}.$$

The condition that $\|\Sigma^{(2)'}\| \geq 1 + \delta$ implies that $d \geq \alpha^{-f_2}$ for some constant $f_2 \geq f_1$. Indeed, otherwise $\Sigma^{(2)'}$ would be "too" close to the set of upper (or lower) triangular unipotents, and by conjugating it with suitable diagonal element it would get too close to 1.

Choose $b, c \in \Sigma$ such that $|(b^2)_{12}|, |(c^2)_{21}| \geq d$. Up to replacing b, c by their inverses we may assume:

$$|\alpha|^{-f_2} \leq |b_{11}|, |b_{21}|, |c_{11}|, |c_{12}| \leq |\alpha|^{f_2}.$$

Denote by $p = [(1, 0)]$, $q = [(0, 1)]$ the attracting and repelling points of a . Since $d \geq |\alpha|^{-f_2}$ it follows that for some f_3 ,

$$d(b \cdot p, \{p, q\}) \geq |\alpha|^{-f_3} \quad \text{and} \quad d(c^{-1} \cdot q, \{p, q\}) \geq |\alpha|^{-f_3}.$$

Thus by multiplying a power of a by b on the left and by c on the right, we get a very contracting element whose attracting and repelling points are bounded away from p and q . In order to ensure that they are not close to each other we may multiply the new element further by a power of a on both sides. This will "move" the attracting and repelling points of the new element close to p and q respectively, but with a distance which can be bounded from below. Hence if we chose a constant n_5 sufficiently large, and then a sufficiently larger constant n_6 and set

$$a' = a^{n_5} x a^{n_6} y a^{n_5},$$

then a, a' form a ping-pong pair, hence generate a free group.

Suppose now that $\text{char}(\mathbb{K}) > 0$. The only difference is that we don't have an analog of Jordan's theorem for finite groups. Again using Lemma 3.8 we may, after replacing Σ by a bounded power of it, assume that $\Sigma^{(2)'}$ in addition to the condition above, does not generate a nilpotent subgroup. Then in case $\langle \Sigma^{(2)'} \rangle$ is infinite, it is Zariski dense and we can proceed as above. On the other hand if $\langle \Sigma^{(2)'} \rangle$ is finite then it is desecrate in any completion k of \mathbb{K} . We can then work in any k in which $\langle \Sigma \rangle$ is unbounded since $\{\Sigma^{(2)'}\}$ cannot be "too" close to a unipotent subgroup, because if it was, the argument of Zassenhouse theorem would imply that it is unipotent. Thus we can apply the same proof as above also in this case. Note that in the positive characteristic the proof does not rely on Theorem 6.2. □

Again, by first order logic, Theorem 7.1 could be formulated in terms of finite groups:

Corollary 7.2. *There is a function f defined on prime powers $q = p^n$ and tending to infinity, such that for any symmetric generating set Σ of $SL_2(\mathbb{F}_q)$ containing the identity, there are two elements $a, b \in \Sigma^{m_2}$ such that the Cayley graph $\chi(SL_2(\mathbb{F}_q), \{a, b\})$ has girth $\geq f(q)$.*

Theorem 7.1 implies the following:

Theorem 7.3. *There is a constant $\epsilon_2 > 0$ such that if $A \subset SL_2(\mathbb{F})$ is a set (not necessarily symmetric) generating a Zariski dense subgroup, and $B \subset SL_2(\mathbb{F})$ is any finite set, then for some $a \in A$ we have*

$$|aB \setminus B| \geq \epsilon_2 |B|.$$

In terms of finite groups, Theorem 7.3 is formulated as:

Corollary 7.4. *Given n, d there is $l = l(n, d)$ such that whenever $q = p^k \geq l$, for any d generators $\gamma_1, \dots, \gamma_d$ of $SL_2(\mathbb{F}_q)$ and any subset A of $SL_2(\mathbb{F}_q)$ of size $\leq n$ there is $1 \leq i \leq d$ such that $\gamma_i \cdot A \setminus A \geq \epsilon_2 |A|$.*

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